# On Solution of the Dubins Touring Problem

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***Abstract*— The Dubins traveling salesman problem (DTSP) combines the combinatorial optimization of the optimal sequence of waypoints to visit the required target locations with the continuous optimization to determine the optimal headings at the waypoints. Existing decoupled approaches to the DTSP are based on an independent solution of the sequencing part as the Euclidean TSP and finding the optimal headings of the waypoints in the sequence. In this work, we focus on the determination of the optimal headings in a given sequence of waypoints and formulate the problem as the Dubins touring problem (DTP). The DTP can be solved by a uniform sampling of possible headings; however, we propose a new informed sampling strategy to find approximate solution of the DTP. Based on the presented results, the proposed algorithm quickly converges to a high-quality solution, which is less than 0.1% from the optimum. Besides, the proposed approach also improves the solution of the DTSP, and its feasibility has been experimentally verified in a real practical deployment.**

## I. Introduction

Curvature-constrained path planning aims to provide a cost efficient path for a non-holonomic vehicle such as Dubins vehicle [1] that models car-like or aircraft vehicles with the minimal turning radius $\rho$ moving with a constant forward velocity. Probably the most utilized problem formulation for surveillance missions with Dubins vehicles is the *Dubins traveling salesman problem* (DTSP) [2], [3] which is a variant of the NP-hard combinatorial optimization TSP. The DTSP stands to find a closed shortest path to visit a given set of target locations in a plane while the path satisfies the motion constraints of Dubins vehicle [4]. The DTSP is also NP-hard [5] as it includes a solution of the TSP; however, it includes additional challenge related to the determination of the optimal heading of the vehicle at each target location. The total tour length depends not only on the sequence of the visits but also on the vehicle heading at each target location.

The difficulty of simultaneous determination of the headings with finding the optimal sequence of the visits motivated researchers to address the DTSP by relaxing the mutual dependency of these two subproblems and assume a sequence of the targets is given, or finite sets of possible headings are assumed in sampling-based approaches [6]. In this paper, we focus on the approach that relies on a given sequence such as, e.g., *Alternating algorithm* (AA) [4], receding horizon methods [7], or *Local iterative optimization* (LIO) [8].

Having the sequence, the problem of determining the optimal headings can be called the *Dubins touring problem* (DTP). Although the DTP can be considered as a subproblem

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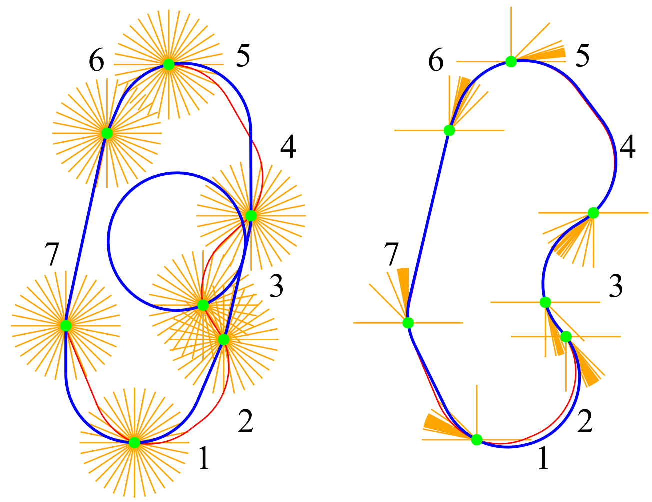


(a) Uniform sampling – $N = 224$, $\mathcal{L} = 19.8$, $\mathcal{L}_U = 13.8$, $t = 128$ ms

(b) Proposed sampling – $N = 128$, $\mathcal{L} = 14.4$, $\mathcal{L}_U = 14.2$, $t = 76$ ms

Fig. 1. A solution of the DTSP for a given sequence of the targets (the green disks) with the total number of samples $N$, final path length $\mathcal{L}$, and lower bound $\mathcal{L}_U$. The found solution is the blue curve, and the red curve is its lower bound determined as a solution of the Dubins interval problem (DIP) with the cost $\mathcal{L}_U$. The uniform sampling utilizes 32 heading values per each target. The required computational time is denoted $t$.

of the DTSP, we believe the DTP is the fundamental building block of routing problems with Dubins vehicle, and thus it deserves a dedicated formulation. For example in the recently introduced *Dubins orienteering problem* (DOP) [9], a solution of the DTP is a part of the target insertion/deletion step, and the solution of the DOP depends on the sum of the collected rewards, and thus it may not necessarily depend only on the final tour length as in the DTSP. Therefore, in this paper, we focus on the solution the DTP to quickly find a high-quality solution with the estimation of its gap to the optimal solution. The presented approach is motivated by practical needs of the robotic competition MBZIRC [10], [11], where a high-quality solution found in the shortest time possible is desirable because of limited time to plan how to quickly collect as many of high rewarding object as possible.

In particular, we investigate a sampling of the headings in the DTP to reduce the number of required samples. Based on the recent results on the so-called *Dubins interval problem* (DIP) [12] utilized to establish a lower bound of the DTP solution, we developed a new informed sampling-based strategy to quickly determine the most promising headings for the optimal solution of the DTP. The proposed approach quickly converges to a solution of high-quality, and it is less computationally demanding than using a uniform sampling of the headings utilized in [13]. The practical influence of the guided sampling is demonstrated in Fig. 1.

## II. Related work

One of the first DTP-based approaches to the DTSP is the AA [4] in which headings are first established for even edges of the sequence by straight line segments, and then, the optimal Dubins maneuvers are determined for the odd edges. Later on, this approach has been improved by a metaheuristic procedure based on a greedy randomized adaptive search [14]. In [15], the same authors consider a distance between two consecutive targets to improve the basic idea of the AA. Only two consecutive targets in the sequence are considered in all these approaches, and thus these algorithms are computationally efficient and their time complexity for $n$ targets can be bounded by $O(n)$.

A look-ahead approach based on more targets in the sequence has been proposed in [7] where three consecutive targets are considered, and the authors report improved results over the simple AA. The idea has been further investigated in [16] for a combination of the $k$-look-ahead technique with the local improvement of the 2-Opt heuristic; however, the authors do not report on the number of utilized headings per each target and also do not report on the computational time.

An optimal solution of the DTP based on the convex optimization has been proposed in [17] for instances where each pair of consecutive targets are more than four times the minimal turning radius apart. The authors reduce the DTP to a family of $n$-dimensional convex optimization sub-problems where the number of sub-problems can be bounded by $2^{2n-2}$.

The aforementioned heuristic approaches provide a solution of the DTP, but none of them provides a tight-lower bound. The first systematic procedure to provide both a solution and its lower bound has been presented in [18] and further evaluated in [13], but without presenting computational requirements. The herein proposed informed sampling strategy directly builds on the results of the lower bound presented in [13] and also on the solution of DIP introduced by the same authors in [12]. The main contribution of the current paper is in the increased computational efficiency by avoiding dense uniform sampling of the headings, and thus a better solution can be found for a limited computational time than for the uniform sampling.

## III. Problem Statement

The *Dubins touring problem* (DTP) stands to determine a shortest curvature-constrained tour to visit a given sequence of $n$ target locations, $\mathcal{P} = (p_1, \ldots, p_n)$, $p_i \in \mathbb{R}^2$. The state $q$ of Dubins vehicle [1] is represented as a triplet $q = (x, y, \theta)$ and $q \in SE(2)$, where $(x, y) \in \mathbb{R}^2$ is the vehicle position in a plane and $\theta \in \mathbb{S}^1$ is the vehicle heading at $(x, y)$. Dubins vehicle is constrained to move only forward at the constant speed $v$ and has the minimum turning radius restricted to $\rho$. The motion of the vehicle can be described as:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = v \begin{bmatrix} \cos\theta \\ \sin\theta \\ \frac{u}{\rho} \end{bmatrix}, \quad |u| \le 1, \tag{1}$$

where $u$ is the control input.

In the DTP for the DTSP, Dubins vehicle is requested to visit a given sequence of $n$ target locations $\mathcal{P}$ and return to the starting location. Since the order of the targets is given, the problem is to find a particular heading for each target while the tour constructed from Dubins maneuvers [1] is the shortest possible. This can be formulated as an optimization problem for $n$ variables representing particular headings $T = \{\theta_1, \ldots, \theta_n\}$ with the piecewise continuous cost function:

$$\mathcal{L}(T, \mathcal{P}) = \sum_{i=1}^{n-1} \mathcal{L}(q_i, q_{i+1}) + \mathcal{L}(q_n, q_1), \tag{2}$$

where $\mathcal{L}(q_i, q_j)$ is the length of the shortest Dubins maneuver between the configurations $q_i$ and $q_j$. The optimization problem can be stated as follows.

*Problem 3.1 (DTP):*

$$\begin{aligned} \textit{minimize}_{\,T} \quad & \mathcal{L}(T, \mathcal{P}) = \sum_{i=1}^{n-1} \mathcal{L}(q_i, q_{i+1}) + \mathcal{L}(q_n, q_1) \\ \textit{subject to} \quad & q_i = (p_i, \theta_i), \quad p_i \in \mathcal{P}, \ i = 1, \ldots, n \end{aligned}$$

The proposed approach for the optimal solution of the DTP is based on a solution of DIP, which is detailed in Section III-A. Therefore, we further distinguish a particular value of the heading $\theta$ and an interval of heading values $\Theta$ in this paper. Moreover, based on the heading intervals we can establish a lower bound of the optimal solution of the DTP, while a feasible solution represents an upper bound.

### A. Dubins Maneuvers and Dubins Interval Problem (DIP)

In [1], Dubins shows that for two states $q_i$ and $q_j$ the optimal path for a vehicle with the minimal turning radius $\rho$ is one of the six possible maneuvers that consist of the straight line segment $S$ and a part of the circle with the radius $\rho$ denoted by the $C$-segment, which is further distinguish based on the orientation of the circle as $L$ and $R$. Each optimal path can have at most three segments (zero length segments are allowed) which provide two types of paths:

- CCC type: LRL, RLR;
- CSC type: LSL, LSR, RSL, RSR.

The optimal path connecting two states is called Dubins maneuver, and it can be computed by a closed-form expression. The optimal path is easy to compute if headings at the locations are prescribed. However, the length of Dubins maneuver is only a piecewise continuous with the discontinuity between CCC and CSC maneuvers, depending on the mutual distance of the states and headings.

A generalization of this simple Dubins planning for intervals of possible headings at the locations instead of a single heading value at the particular state has been proposed in [12]. The authors call the problem as the *Dubins interval problem* (DIP), and it is detailed in the next paragraph.

DIP stands to find the shortest Dubins maneuver for two locations $p_i$ and $p_j$ for which the departure angle from $p_i$ is in the given interval $\theta_i \in \Theta_i$ and the arrival angle at $p_j$ must be in $\theta_j \in \Theta_j$, where $\Theta_i = [\theta_i^{min}, \theta_i^{max}]$ and $\Theta_j = [\theta_j^{min}, \theta_j^{max}]$, see Fig. 2. The authors of [12] provide

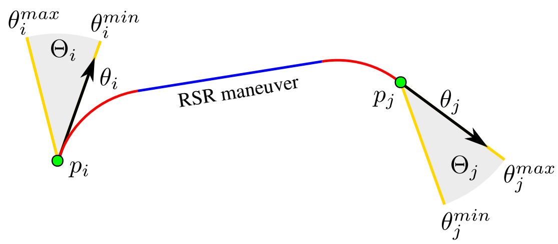


Fig. 2. An instance of the Dubins interval problem to connect $p_i$ and $p_j$ using the departure angle $\theta_i \in \Theta_i$ and the arrival angle $\theta_j \in \Theta_j$

a list of all possible Dubins maneuvers for particular cases of the departure and arrival angles. The types of Dubins maneuvers $S$, $R$, and $L$ are further classified as special cases of $R$ and $L$ if parts of the circling maneuver are longer (in an angular distance) than $\pi$ and they are called as $R_\psi$ and $L_\psi$, respectively. The optimal solution of DIP is one of the following nine maneuvers according to the particular values of the headings $\theta_i \in \Theta_i$ and $\theta_j \in \Theta_j$:

1) S or L$_\psi$ or R$_\psi$ [1]
2) LSR for $\theta_i = \theta_i^{max}$ and $\theta_i = \theta_j^{max}$;
3) LSL or LR$_\psi$L for $\theta_i = \theta_i^{max}$ and $\theta_j = \theta_j^{min}$;
4) RSL for $\theta_i = \theta_i^{min}$ and $\theta_j = \theta_j^{min}$;
5) RSR or RL$_\psi$R for $\theta_i = \theta_i^{min}$ and $\theta_j = \theta_j^{max}$;
6) LS or LR$_\psi$ for $\theta_i = \theta_i^{max}$ and $\theta_j \in \Theta_j$;
7) RS or RL$_\psi$ for $\theta_i = \theta_i^{min}$ and $\theta_j \in \Theta_j$;
8) SR or L$_\psi$R for $\theta_i \in \Theta_i$ and $\theta_j = \theta_j^{max}$;
9) SL or R$_\psi$L for $\theta_i \in \Theta_i$ and $\theta_j = \theta_j^{min}$.

The particular optimal maneuver can be selected regarding the shortest Dubins maneuver for these nine cases, which is a bit more complex than a solution of the original Dubins maneuver for single heading values, but the optimal solution of DIP is still determined by a closed-form expression.

Notice, if we allow a full range of heading values, i.e., $\Theta_i = \Theta_j = [0, 2\pi)$, the solution is the CSC maneuver with the zero length circle parts and the length of the straight line segment equals to the Euclidean distance between $p_i$ and $p_j$.

## IV. Proposed Sampling Strategy for the DTP

Based on the analysis and solution of DIP, we propose a new iterative sampling-based algorithm to solve the DTP. The key idea of the proposed approach is to sample heading at the targets by the informed way using a lower bound of the DTP [18]. The lower bound solution is utilized for determining the promising candidates of the heading intervals. Such candidate intervals are iteratively refined, and the process is repeated until a selected angular resolution $\epsilon$ is reached or after a finite number of refinements. Moreover, since a more precise estimation of the lower bound is determined at each iteration, the iterative refinement can be terminated once the ratio of the length of the found DTP solution to the value of the lower bound reaches the requested approximation ratio $\alpha$. Particular components of the proposed algorithm are detailed in the following paragraphs.

[1] Authors of [12] claimed that L$_\psi$R$_\psi$ or R$_\psi$L$_\psi$ could also be the optimal solution of DIP, but it is not necessary to consider this case because this is not locally optimal in any instance of DIP.

### A. Heading Intervals

The lower bound of the DTP is computed from a solution of DIP for which the heading values at the target locations are constrained to particular intervals [18]. Thus, for each target location $p_i \in P$, a set of heading intervals $H_i$ is maintained. $H_i$ splits the whole range of possible heading values $\theta_i \in [0, 2\pi)$ into $k_i$ not necessarily equally sized heading intervals $H_i = \{[\theta_i^1, \theta_i^2], [\theta_i^2, \theta_i^3], \ldots, [\theta_i^{k_i}, \theta_i^1]\}$. For a better readability, we denote the symbol $\Theta_i^l$ to a particular heading interval $\Theta_i^l = [\theta_i^l, \theta_i^{l+1}]$. Hence, for each target location $p_i$, the set of heading intervals is $H_i = \{\Theta_i^1, \ldots, \Theta_i^{k_i}\}$. Having this notation, we consider the problem of finding a solution of the DTP as a problem to efficiently create a set of particular heading intervals $\mathcal{H} = \{H_1, \ldots, H_n\}$.

### B. Lower Bound Solution of the DTP

For the heading intervals $\mathcal{H}$ with up to $k$ intervals per each target, the lower bound of the DTP solution is determined by computing an optimal solution of the corresponding DIP [18]. Since the number of intervals is finite, it is possible to create an oriented graph with $n$ layers. Each layer corresponds to one target location $p_i$ and consists of $k_i$ nodes, each for one heading interval in $H_i$. The graph nodes are connected by edges representing a solution of DIP, i.e., the weight associated with each edge is the length of the solution of DIP. The graph is visualized in Fig. 3.

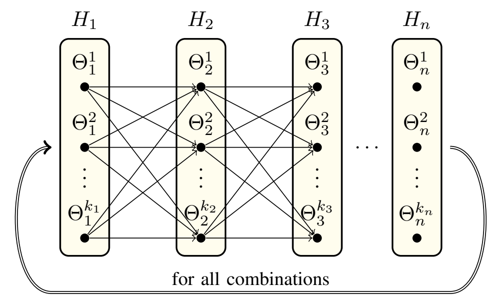


Fig. 3. An example of the search graph for the Dubins touring problem

In this graph, the lower bound $\mathcal{L}_U(\mathcal{P}, \mathcal{H})$ is determined for the current heading intervals $\mathcal{H}$ using a feed-forward search evaluating all possible paths from the starting target $p_1$ with up to $k$ headings per each target in the sequence $\mathcal{P}$. Since the problem contains $n$ target locations with up to $k$ heading intervals for each target, the graph can have up to $nk^2$ edges. Therefore, the time complexity to find the shortest tour in the graph can be bounded by $O(nk^3)$. Notice, if a single heading value is used for each heading interval $\theta_i^l \in \Theta_i^l$ for $1 \le i \le n$ and $1 \le l < k_i$, the same procedure can be used to find a feasible solution of the DTP.

### C. Refinement of the Heading Intervals

The current lower bound solution is a correct estimation of the minimum tour length; however, it is not necessarily a feasible solution. In fact, the lower bound solution is rarely feasible because of a discontinuity in the heading at each target. Therefore, we introduce the *angular resolution* $\epsilon$ to denote the size of the intervals which directly limits the maximal discontinuity of the vehicle heading.

The algorithm starts with a relatively high $\epsilon$ and iteratively refines promising intervals to avoid dense uniform sampling, and thus decrease the computational burden. Having a lower bound solution of the DTP for the current $\mathcal{H}$, the algorithm splits only the intervals presented in the current lower bound solution without losing candidate headings presented in other intervals. By repeating this procedure, the length of the lower bound solution can increase and, eventually, it converges to the optimal solution. Moreover, a single heading value can be sampled for each interval and determine a feasible solution of the DTP for the current $\mathcal{H}$ by the same procedure as the determination of the lower bound.

Notice that even though the solution of DIP may cause discontinuities in particular heading values at the waypoints, any heading value from the interval can be selected and fixed for solving the DTP. Therefore, a feasible solution of the DTP for such fixed heading values is always determined.

### D. Proposed Informed Sampling Algorithm for the DTP

The proposed algorithm for solving the DTP is directly based on the aforementioned refinement procedure denoted by $\mathrm{refineDTP}(\mathcal{P}, \epsilon, \mathcal{H})$, which refines $\mathcal{H}$ up to the angular resolution $\epsilon$. Then, for the refined $\mathcal{H}$, a feasible solution is found by the $\mathrm{solveDTP}(\mathcal{P}, \mathcal{H})$ procedure, i.e., using the forward search graph in Fig. 3. The value of $\epsilon$ is gradually decreased to the requested $\epsilon_{req}$, and for each $\epsilon$, both the lower and upper bound solutions are determined. Hence, the proposed algorithm has any-time property, and an updated feasible solution is available at the end of each iteration. The refinement procedure is summarized in Algorithm 1.

**Algorithm 1:** Proposed Iterative Algorithm for the DTP

**Input**: $P$ – Target locations to be visited
**Input**: $\epsilon_{req}$ – Requested angular resolution
**Input**: $\alpha_{req}$ – Requested quality of the solution
**Output**: $T$ – A tour visiting the targets

$\epsilon \leftarrow 2\pi$ `// initial angular resolution`
$\mathcal{H} \leftarrow \mathrm{createIntervals}(P, \epsilon)$ `// initial intervals`
$\mathcal{L}_L \leftarrow 0$ `// init lower bound`
$\mathcal{L}_U \leftarrow \infty$ `// init upper bound`
**while** $\epsilon > \epsilon_{req}$ and $\mathcal{L}_U/\mathcal{L}_L > \alpha_{req}$ **do**
  $\epsilon \leftarrow \epsilon/2$
  $(\mathcal{H}, \mathcal{L}_L) \leftarrow \mathrm{refineDTP}(\mathcal{P}, \epsilon, \mathcal{H})$
  $(T, \mathcal{L}_U) \leftarrow \mathrm{solveDTP}(\mathcal{P}, \mathcal{H})$
**end**
**return** $T$

The algorithm is terminated after reaching the requested angular resolution $\epsilon_{req}$, which guarantees a termination after a finite number of iterations because there is a finite number of the heading intervals for $\epsilon_{req}$. Alternatively, the refinement can be terminated when the solution quality of the feasible solution is below $\alpha_{req}$. However, this may not properly stop the algorithm in a reasonable time for very small $\alpha_{req}$ if the solution requires very fine sampling. Therefore, it is convenient to combine both conditions together. If the anytime property of the algorithm is utilized, the refinement loop can be terminated after a given computational time as the first solution is found in a few milliseconds.

## V. RESULTS

The performance of the proposed algorithm has been evaluated in randomly generated instances of the DTP and compared with the uniform sampling strategy to verify if the sampling strategy guided by the solution of the lower bound of the DTP provides better solutions with lower computational requirements than the uniform sampling. Then, the proposed DTP solver has been deployed in solving the DTSP for a given sequence of visits to the targets and compared with the existing heuristics for the DTSP in instances with increasing number of target locations $n$. For brevity and without loss of generality, we consider Dubins vehicle with $v = 1$ m.s$^{-1}$ and $\rho = 1$ m in the evaluated problems. Finally, a feasibility of the solution found by the proposed solver has been verified in an experimental deployment with the real vehicle that follows the planned path. All the evaluated algorithms have been implemented in C++, and the presented results have been obtained using a single core of the AMD Phenom[tm] II X6 1090T CPU running at 3.2 GHz, and thus the required computational times can be directly compared.

### A. Computational Requirements of the DTP Solvers

The DTP can be directly used in the solution of the DTSP, and therefore, we studied computational requirements of the sampling-based solvers of the DTP for randomly generated DTSP instances for which the sequence of the targets is determined as an optimal solution of the Euclidean TSP found by Concorde [19], e.g., similarly as for the AA approach [4]. It is known that a solution of the DTSP and also DTP depends on the mutual distance of the targets with respect to the minimal turning radius $\rho$. Therefore, we generate random instances of the DTSP according to the relative density of the targets $d$ inside an area with the dimensions $s \times s$, where $s$ is determined as $s = (\rho\sqrt{n})/d$. Due to the limited space, $d = 0.5$ is considered in the presented results, and 20 random instances are created for $n \in \{10, 20, 50, 70, 100\}$, which gives 100 instances in total.

The real computational requirements of the proposed algorithm are mostly related to the number of headings for the current intervals $\mathcal{H}$. It can be expected that for a finer angular resolution or a low requested ratio $\alpha$, the computational time will increase. Therefore, we consider the any-time property of the algorithm and evaluate its computational requirements as the quality of found solutions $\alpha$ for the given computational time. The value of $\alpha$ is the ratio of the feasible path length $\mathcal{L}$ and the length $\mathcal{L}_u$ of the lower bound solution, $\alpha = \mathcal{L}/\mathcal{L}_U$.

Summarized results are shown in Fig. 4, where the solution quality is presented as average values from 20 trials of the sequences with 50 target locations and the error bars denote the standard deviations. The results indicate the proposed

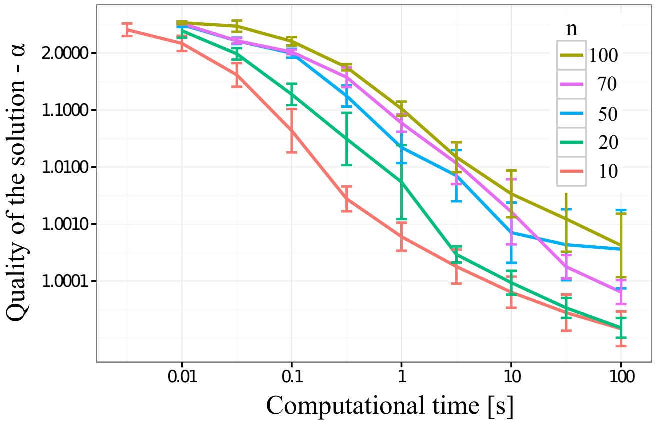


Fig. 4. Average solution quality $\alpha$ computed as the ratio of the solution cost to its lower bound cost determined by the proposed algorithm according to the given computational time. Notice both axes are in log-scale.

algorithm is able to find solutions that are less than 0.1% from the optimal solution (lower bound) in about 10 seconds. Further improvement of the solution for more computational time can be observed; however, from a practical point of view, a solution closer than 0.01% from the optima might be considered as the optimum.

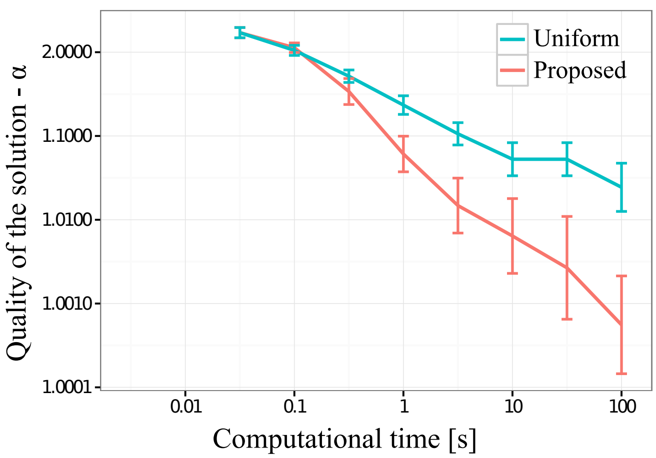


Fig. 5. Average solution quality $\alpha$ according to the given computational time. The results are average values of 20 trials of the DTP instances with $n = 50$ targets and the relative density $d = 0.5$. Notice both axes are in log-scale.

The proposed approach has been compared with the uniform sampling utilized in [13], where the authors use 128 samples of the heading values per each target location to uniformly split possible heading intervals (two times each step). Since a solution of the DTP with 128 samples per each of 50 targets can be computationally demanding, we incrementally increase the number of samples (two times each step) and solve the problem for each individual number of uniformly distributed headings up to the resolution for which the solution of the DTP is found in less than 100 seconds. The results of the evaluation depicted in Fig. 5 indicate that even though the uniform sampling with a high number of samples provides high-quality solutions, from the practical point of view, the proposed refinement converges to the optimal solutions much faster. In particular, the proposed refinement strategy provides solutions in less than 10% from the optimal solution in less than one second, while the uniform sampling needs about 10 seconds. Moreover, the proposed refinement strategy is capable of providing solutions that are less than 1% from the optima in less than 10 seconds on the utilized computer, and it is even capable of providing solutions that are less than 0.1% from the optima. To find such high-quality solutions with uniform sampling, a high number of samples is needed, and the computational requirements are significantly higher. Therefore, for high-quality solutions, e.g., in solving the Orienteering problem [9] where the evaluation of the Dubins tour is used to add or remove particular targets to/from the current reward collecting tour, it is preferable to utilize the proposed refinement strategy.

### B. Performance of the Proposed Algorithm in the DTSP

The proposed algorithm has been compared with the existing heuristic solvers for the DTSP, where the sequence of the visits to the targets is determined by a solution of the underlying Euclidean TSP. We consider the same problems as in the previous evaluation and the AA [4] and LIO [8] heuristic algorithms for this evaluation. In addition, the Memetic algorithm for the DTSP [20] is considered to evaluate an influence of the high-quality solution of the DTP provided by the proposed algorithm with the DTSP algorithm that does not rely on the sequence of the visits to the targets.

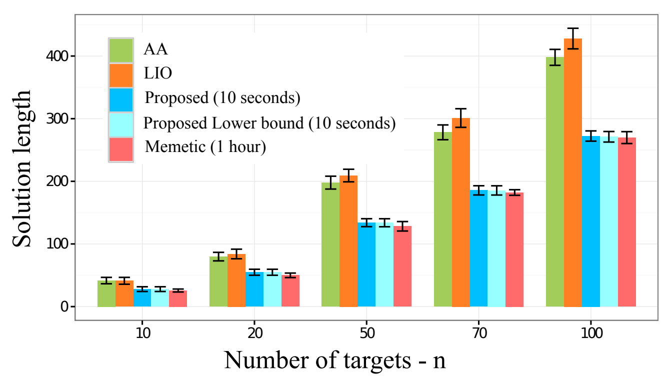


Fig. 6. Average solution length (from 20 trials) for the DTSP instances with $n$ targets and $d = 0.5$. The computational time for the proposed algorithm has been restricted to 10 seconds and for the Memetic algorithm to 1 hour.

The heuristic algorithms AA and LIO found solutions in less than few seconds for a given sequence, and the solutions are not further improved if more computational time is available. On the other hand, the Memetic algorithm can provide high-quality solutions at the cost of high computational requirements. Therefore, we restrict its computational time to 1 hour per each trial and use a computational grid (with the Intel Xeon CPUs). The results are presented in Fig. 6.

It can be observed that the proposed algorithm provides the best solution of the underlying DTP of the DTSP instances. Regarding the lower bound and the feasible solution provided by the proposed algorithm, the found solutions are very close to the optimal solution of the DTP, albeit the computational time of the proposed algorithm has been limited to 10 seconds. Moreover, quite surprisingly, the solutions provided by the proposed DTP algorithm are very close to the solutions provided by the computationally very demanding Memetic algorithm, which in addition optimizes the sequence of the visits to the targets. This is a source of motivation to employ the proposed DTP solver in the DTSP to further improve the solution once a sequence is determined, especially for instances with high values of the targets density $d$.

### C. Real Experiment

A feasibility of the found solution has been experimentally verified with the hexacopter Unmanned Aerial Vehicle (UAV) with forwarding velocity limited to $v = 4$ m.s$^{-1}$, the minimal turning radius $\rho = 6$ m, and limited acceleration $a_{max} = 2.67$ m.s$^{-2}$. The scenario is motivated by a visual inspection of objects of interest that have to be captured by a downward-looking camera in the MBZIRC competition [10]. A visualization of the real deployment is shown in Fig. 7, where the objects of interest can be seen as small light regions inside the disk-shaped target locations with the diameter corresponding to the field of view of the used camera.

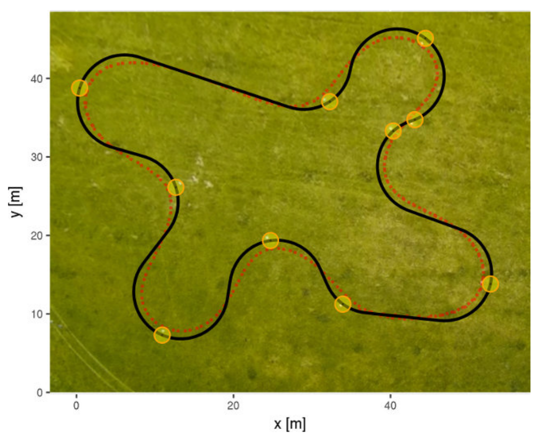


Fig. 7. Planned path (in black) and real executed path captured by the DGPS (in red) of the hexacopter UAV in the DTSP problem with 10 targets

## VI. Conclusion

In this paper, we investigate the Dubins touring problem (DTP) as the fundamental building block of routing problems with Dubins vehicle. The DTP is an important subproblem of the DTSP approaches that rely on a sequence of visits to the targets. A new informed sampling-based algorithm for the DTP has been proposed. It uses an iterative refinement of the possible heading intervals where the optimal headings can be found, and it provides high-quality solutions of the DTP. The proposed algorithm utilizes a tight lower bound of the DTP to guide sampling of the suitable heading intervals, and thus the algorithm can provide a quality guarantee of the found solution. The presented results support the feasibility and quick convergence of the proposed algorithm. Moreover, the comparison with the Memetic algorithm indicates that a near-optimal solution of the DTP can significantly improve a solution of the DTSP, albeit a single sequence is utilized in the DTP-based approaches. Based on the presented results, our further work aims to use the proposed algorithm in a solution of the DTSP, where it can provide a quick evaluation of the candidate sequences of the visits to the targets.

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## References


[1] L. E. Dubins, "On curves of minimal length with a constraint on average curvature, and with prescribed initial and terminal positions and tangents," *American Journal of Mathematics*, pp. 497–516, 1957.

[2] K. J. Obermeyer, "Path planning for a uav performing reconnaissance of static ground targets in terrain," in *AIAA Guidance, Navigation, and Control Conference*, 2009, pp. 10–13.

[3] P. Oberlin, S. Rathinam, and S. Darbha, "Today's traveling salesman problem," *Robotics & Automation Magazine, IEEE*, vol. 17, no. 4, pp. 70–77, 2010.

[4] K. Savla, E. Frazzoli, and F. Bullo, "On the point-to-point and traveling salesperson problems for Dubins' vehicle," in *American Control Conference*, 2005, pp. 786–791.

[5] J. Le Ny, E. Feron, and E. Frazzoli, "On the Dubins Traveling Salesman Problem," *IEEE Transactions on Automatic Control*, vol. 57, no. 1, pp. 265–270, 2012.

[6] K. J. Obermeyer, P. Oberlin, and S. Darbha, "Sampling-based path planning for a visual reconnaissance unmanned air vehicle," *Journal of Guidance, Control, and Dynamics*, vol. 35, no. 2, pp. 619–631, 2012.

[7] X. Ma and D. A. Castanon, "Receding horizon planning for Dubins traveling salesman problems," in *45th IEEE Conference on Decision and Control*, 2006, pp. 5453–5458.

[8] P. Váňa and J. Faigl, "On the dubins traveling salesman problem with neighborhoods," in *IROS*, 2015, pp. 4029–4034.

[9] R. Pěnička, J. Faigl, P. Váňa, and M. Saska, "Dubins orienteering problem," *Robotics and Automation Letters*, vol. 2, no. 2, pp. 1210–1217, 2017.

[10] "MBZIRC team of the Czech Technical University," [cited 17 Mar 2017]. [Online]. Available: http://mrs.felk.cvut.cz/projects/mbzirc

[11] "Mohamed Bin Zayed International Robotics Challenge (MBZIRC)," [cited 17 Mar 2017]. [Online]. Available: http://www.mbzirc.com

[12] S. Manyam, S. Rathinam, D. Casbeer, and E. Garcia, "Shortest Paths of Bounded Curvature for the Dubins Interval Problem," *arXiv preprint arXiv:1507.06980*, 2015.

[13] S. Manyam, S. Rathinam, and D. Casbeer, "Dubins paths through a sequence of points: Lower and upper bounds," in *ICUAS*, 2016, pp. 284–291.

[14] D. G. Macharet, A. A. Neto, V. F. da Camara Neto, and M. F. Campos, "Nonholonomic path planning optimization for dubins' vehicles," in *ICRA*, 2011, pp. 4208–4213.

[15] D. G. Macharet and M. F. Campos, "An orientation assignment heuristic to the dubins traveling salesman problem," in *Advances in Artificial Intelligence–IBERAMIA*, 2014, pp. 457–468.

[16] P. Isaiah and T. Shima, "Motion planning algorithms for the Dubins Travelling Salesperson Problem," *Automatica*, vol. 53, pp. 247–255, 2015.

[17] X. Goaoc, H.-S. Kim, and S. Lazard, "Bounded-curvature shortest paths through a sequence of points using convex optimization," *SIAM Journal on Computing*, vol. 42, no. 2, pp. 662–684, 2013.

[18] S. Manyam and S. Rathinam, "A Tight Lower Bounding Procedure for the Dubins Traveling Salesman Problem," *arXiv preprint arXiv:1506.08752v2*, 2015.

[19] D. Applegate, R. Bixby, V. Chvátal, and W. Cook, "CONCORDE TSP Solver," 2003, [cited 14 May 2017]. [Online]. Available: http://www.tsp.gatech.edu/concorde.html

[20] X. Zhang, J. Chen, B. Xin, and Z. Peng, "A memetic algorithm for path planning of curvature-constrained uavs performing surveillance of multiple ground targets," *Chinese Journal of Aeronautics*, vol. 27, no. 3, pp. 622–633, 2014.