Bounding Optimal Headings in the Dubins Touring Problem

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ABSTRACT

The Dubins Touring Problem (DTP) stands to find the shortest curvature-constrained multi-goal path connecting a prescribed sequence of locations. The problem is to determine the optimal vehicle heading angle at each location and thus find the shortest sequence of Dubins paths. The heading angles can be determined by iterative refinement of possible heading intervals for which finer resolution yields a shorter path at the cost of increased computational requirements. In this paper, we introduce a novel method to bound the optimal heading angles by eliminating unpromising intervals that cannot contribute to the optimal solution. The method is employed in the branch-and-bound solution of the DTP, where it significantly reduces the search space in finding the optimal solution.

KEYWORDS

Dubins vehicle, Curvature-constrained path, Dubins Interval Problem, Branch-and-bound algorithm

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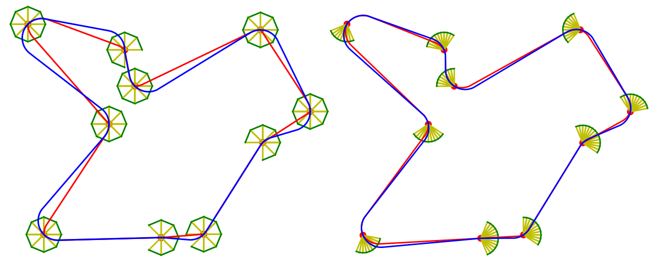

Figure 1: An instance of the DTP with bounds on the optimal headings computed by the BnB algorithm. The red disks represent the given locations. The heading intervals, shown in yellow, has a particular angular resolution $\omega_{\mathbf{max}}$ and some of them are bounded. A solution for $\omega_{\mathbf{max}} = 8$ is shown on the left. For $\omega_{\mathbf{max}} = 32$, a solution visualized on the right has the relative optimality gap between the feasible path (in blue) and the lower bound path (in red) is 1.43 %.

1 INTRODUCTION

The studied *Dubins Touring Problem* (DTP) [5] is a multi-goal generalization of the curvature-constrained path planning for a vehicle with a limited turning radius. The shortest curvature-constrained path between two points with the prescribed heading angles of the vehicle was addressed by Dubins in 1975 [4]. Such a point-to-point optimal (Dubins) path can be found by a closed-form expression, and it consists of three arc types: right (R) and left (L) parts of the circle and a straight line segment (S). However, we need to determine the heading angle at each location for a multi-goal path with a given sequence of locations to be visited. Once the heading angles are determined, the final path is found by computing optimal (point-to-point) Dubins paths between the consecutive locations in the sequence. Hence, the DTP can be seen as a continuous optimization problem to find the most suitable heading angle at each location to minimize the length of the final multi-goal Dubins path.

The DTP is inherently included in the combinatorial optimization of the *Dubins Traveling Salesman Problem* (DTSP) [12] for which several heuristic approaches can be found in the literature. The first approach is the *Alternating Algorithm* (AA) [12] that determines the heading angles by alternating straight segments with Dubins paths. The AA have been improved using a greedy randomized adaptive search metaheuristic [10] and orientation assignment heuristic [9]. A look-ahead alternative based on three consecutive locations has been proposed in [8]. Probably the first optimal solution to the DTP is based on the reduction to 2^{2n-2} convex optimization problems for a sequence of n locations satisfying the so-called D4 assumption, where two consecutive locations are at least four times the minimum turning radius apart [6]. The D4 assumption has been exploited in [13] to address the generalized DTSP with convex goal regions. Besides, the homotopy concept has been utilized in [1, 15].

Besides the approaches above, the most relevant work to the studied DTP is the *Dubins Interval Problem* (DIP) that stands to find the shortest Dubins path between two locations with prescribed heading angle intervals [11]. The DIP enables computation of a tight lower bound on the optimal cost of the DTP using sampled heading intervals. Tight lower bounds need a fine angular resolution that is demanding for uniform sampling [11] that can be addressed by iterative refinement of the intervals [5]. However, the intervals are only incrementally divided in both methods without any mechanism to prune non-perspective intervals.

In the present work, we propose a novel method to eliminate unpromising heading intervals and thus bound the optimal heading angles in the *Branch-and-Bound* (BnB) solution of the DTP. The bounding is based on the newly introduced maximization variant of the DIP (Max-DIP) to determine the longest Dubins path connecting two locations with the defined intervals of the heading angles.

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2 THE DUBINS TOURING PROBLEM

The studied DTP stands to find the shortest path for the Dubins vehicle going through a sequence of n locations $\mathcal{P} = (p_1, \ldots, p_n)$, $p_i \in \mathbb{R}^2$. The Dubins vehicle is assumed to go forward at the constant speed v, and its movement is constrained by its minimum turning radius ρ. The vehicle state $q = (x, y, \theta)$, $q \in SE(2)$ consists of its position $(x, y) \in \mathbb{R}^2$ and heading angle $\theta \in \mathbb{S}^1$. The heading is controlled by the input u and the state can be expressed as

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = v \begin{bmatrix} \cos\theta \\ \sin\theta \\ \frac{u}{\rho} \end{bmatrix}, \quad |u| \leq 1. \tag{1}$$

The DTP can be formulated as a continuous optimization Problem 2.1 to determine the configurations $Q = \{q_1, \ldots, q_n\}$ such that the total length of the path is minimized, with $\mathcal{L}(q_i, q_j)$ as the length of the optimal Dubins path from q_i to q_j satisfying (1).

Problem 2.1 (Dubins Touring Problem (DTP)).

$$\begin{aligned} & \mathit{minimize}_Q && \sum_{i=1}^{n-1} \mathcal{L}(q_i, q_{i+1}) + \mathcal{L}(q_n, q_1) \\ & \text{subject to} && q_i = (p_i, \theta_i),\ p_i \in \mathcal{P},\ i = 1, \ldots, n. \end{aligned}$$

In this paper, we focus on the optimal solution of the DTP using the BnB. Although the BnB solution to the DTP can be straightforward to formulate, a practical solution needs to address computational requirements; otherwise, the solution would be computationally intractable even for relatively small instances. Therefore, it is necessary to have a mechanism to bound intervals of heading angles by determining what intervals would surely not contribute to the optimal solution of the DTP.

3 BOUNDING OPTIMAL HEADING ANGLES

The proposed BnB solution to the DTP builds upon ideas of [5], where possible heading angles are divided into angular intervals to compute a tight lower bound using the DIP [11]. Further, the intervals are gradually refined in [5] to tighten the bound, where the number of intervals iteratively grows. Therefore, we propose a new upper bound scheme to bound the optimal headings and remove intervals that cannot further contribute to the optimal solution. The new upper bound is based on the so-called *Maximization Dubins Interval Problem* (Max-DIP) that is detailed in this section.

The used BnB for the DTP starts with the angular resolution $\omega_{\text{init}} = 4$ for creating initial heading intervals $\mathcal{H}$ that are initially divided uniformly. The resolution is then iteratively increased (as in [5]), which represents the branching of possible intervals using lower bounds on the optimal solution based on the DIP. On the other hand, unpromising intervals are removed using the proposed bounding based on upper bound values of the optimal solution computed from the Max-DIP. The iterative BnB-based solution is summarized in Algorithm 1, and an example of the solution with removed heading intervals is visualized in Fig. 1.

Algorithm 1: BnB-based Solution to the DTP($\mathcal{P}, \omega_{\max}$)

$\omega \leftarrow \omega_{\text{init}}$ `// Initial angular resolution`
$\mathcal{H} \leftarrow$ `SampleIntervals(`$\mathcal{P}$`)` `// Initial intervals`
while $\omega \leq \omega_{max}$ **do**
  $\mathcal{H} \leftarrow$ `BoundIntervals(`$\mathcal{H}$`);`
  $\omega \leftarrow 2\omega$ `// Increase of the angular resolution;`
  $\mathcal{H} \leftarrow$ `RefineIntervals(`$\mathcal{H}$`)`
$Q \leftarrow$ `RetrieveFeasibleSolution(`$\mathcal{H}$`)`

3.1 Lower Bound – DIP

The DIP [11] can be used to determine the lower bound value of the optimal solution for a given sub-sequence with prescribed intervals of the heading angles. Formally, the DIP, further denoted as lower bound $\mathcal{L}_L$, is a continuous optimization to find the shortest Dubins path between two locations p_i and p_j while the heading angles are from the given heading intervals $\theta_i \in \Theta_i$ and $\theta_j \in \Theta_j$.

Problem 3.1 (Dubins Interval Problem (DIP) – $\mathcal{L}_L$).

$$\mathcal{L}_L(\Theta_i, \Theta_j) = \min_{\theta_i \in \Theta_i,\ \theta_j \in \Theta_j} \mathcal{L}((p_i, \theta_i), (p_j, \theta_j)), \tag{2}$$

where $\mathcal{L}(q_i, q_j)$ is the length of the Dubins path from q_i to q_j.

3.2 Maximization Variant of the DIP – Max-DIP

The Max-DIP is a variant of the DIP, where the minimization of the cost function given by the length of the Dubins path is replaced by the maximization. Thus, the Max-DIP is to find the optimal Dubins path from one configuration to another, both with the prescribed heading intervals, such that it is of the maximal length. In general, the length of the Dubins path $\mathcal{L}$ is a piecewise-continuous function. Therefore, in this paper, we limit the Max-DIP for the D4 assumption for which $\mathcal{L}$ becomes a periodic continuous function with a single local maximum.

Assumption 3.1 (D4 condition on mutual distances.). *Two consecutive locations p_i and p_j are at least $4\,\rho$ apart.*

$$d = \frac{\|p_j - p_i\|}{\rho}, \quad d \leq 4, \tag{3}$$

where $\|\cdot\|$ stands for the Euclidean distance.

Furthermore, for using the upper bound in the BnB, we define two variants of the Max-DIP, each with a fixed one of the heading angles. The variant with the fixed initial heading angles is denoted $\mathcal{L}_U^+$, and the variant with the fixed final angle is denoted $\mathcal{L}_U^-$. The variants are defined as Problem 3.2 and Problem 3.3 that are visualized in Fig. 2.

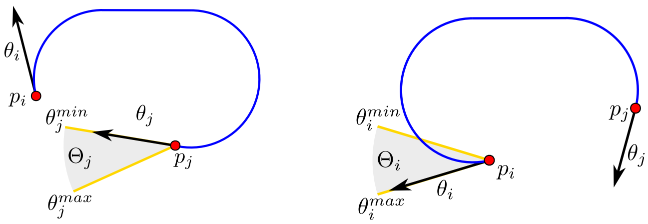
Problem 3.2 (Max-DIP with fixed initial angle ($\mathcal{L}_U^+$)).

$$\mathcal{L}_U^+(\theta_i, \Theta_j) = \max_{\theta_j \in \Theta_j} \mathcal{L}((p_i, \theta_i), (p_j, \theta_j)). \tag{4}$$

Problem 3.3 (Max-DIP with fixed final angle ($\mathcal{L}_U^-$)).

$$\mathcal{L}_U^-(\Theta_i, \theta_j) = \max_{\theta_i \in \Theta_i} \mathcal{L}((p_i, \theta_i), (p_j, \theta_j)). \tag{5}$$

There exists at most one unbounded local maximum for $\mathcal{L}_U^+$ or $\mathcal{L}_U^-$, and thus the global maximum can be found using local optimization. The only exception is the transition between the Dubins paths of the LSR and RSL type, containing straight (S), left (L), and right (R) segments. However, such a case can be detected, and an appropriate solution can be selected.



(a) $\mathcal{L}_U^+$ (b) $\mathcal{L}_U^-$

Figure 2: Example solutions of the proposed Max-DIP with fixed initial (left) and final (right) heading angle.

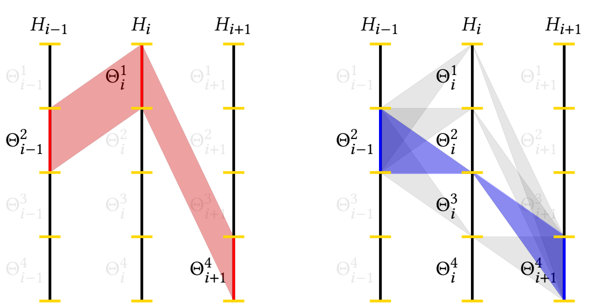
Note that the correctness of the BnB-based solution to the DTP can be guaranteed using the universal upper bound (6) of [3] for a case the D4 assumption is violated.

$$\mathcal{L}_U^{\mathrm{D4+}} = 2\,\pi + \|p_2 - p_1\|. \tag{6}$$

3.3 Bounding Heading Intervals

Determination of the unpromising bound intervals that cannot contribute to the optimal solution is based on the lower bound $\mathcal{L}_L$ and upper bounds $\mathcal{L}_U^+$, $\mathcal{L}_U^-$ that can be computed for any pair of heading intervals. The proposed method works with a sub-sequence of the locations in the given sequence as follows.

First, let assume there is a window of w connections both before and after the location for which we examine a particular heading interval. Hence, the sub-sequence contains $2w$ connections connecting $2w + 1$ locations, and the examined heading interval is always for the location in the middle. An example of the search structure for detecting heading intervals to be removed is depicted in Fig. 3.



(a) Lower bound for $w = 1$ (b) Upper bound for $w = 1$

Figure 3: Search structures for finding bounds of the path visiting the locations $\{p_{i-w}, \ldots, p_i, \ldots, p_{i+w}\}$ for the window size w, $w = 1$ here. The heading interval Θ_i^1 is examined based on the initial interval Θ_{i-w}^2 and the final interval Θ_{i+w}^4. The lower bound solution goes through the examined interval Θ_i^1, but the upper bound is computed independently on Θ_i^1.

Now, let p_i be the location for examining the heading intervals. The lower and upper bounds are computed based on the sub-path from p_{i-w} to p_{i+w} using all the intermediate locations. The lower bound $\mathcal{L}_L$ between two consecutive configurations with particular heading intervals is computed using the DIP. Then, the lower bound for a path going through the intermediate locations with the heading intervals H^x is computed using dynamic programming

$$\mathcal{L}_L(\Theta^a, \Theta^b) = \min_{\Theta^x \in H^x} \left(\mathcal{L}_L(\Theta^a, \Theta^x) + \mathcal{L}_L(\Theta^x, \Theta^b) \right). \tag{7}$$

For the upper bound, we need to evaluate two variants of the Max-DIP. The upper bound between two configurations is computed such that the costs of the first connection layer for p_{i-w} are computed based on $\mathcal{L}_U^+$, see Fig. 3. The final heading angles are fixed and selected as the minimum heading angle from the terminal heading interval. The intermediate connection layers are computed based on the length of the Dubins path using the exact discretization. Using the discrete set of heading angles enables us to find tight upper bounds while maintaining their feasibility. The last layer for p_{i+w} utilizes the $\mathcal{L}_U^-$ variant of the Max-DIP. The upper bound $\mathcal{L}_U$ is then computed as the shortest path given the search graph

$$\mathcal{L}_U(\Theta_{i-w}^a, \Theta_{i+w}^b) = \min_{\Theta_{i-w+1},\ldots,\Theta_{i+w-1}} \Bigg[\mathcal{L}_U^+(\Theta_{i-w}^a, \Theta_{i-w+1}) + \\ + \sum_{j=i-w+1}^{i+w-2} \Big(\mathcal{L}(\Theta_j, \Theta_{j+1}) \Big) + \mathcal{L}_U^-(\Theta_{i+w-1}, \Theta_{i+w}^b) \Bigg], \tag{8}$$

where the heading intervals $\Theta_{i-w+1}, \ldots, \Theta_{i+w-1}$ are selected from the corresponding sets of heading intervals $H_{i-w+1}, \ldots, H_{i+w-1}$. Having lower $\mathcal{L}_L$ and upper $\mathcal{L}_U$ bounds for an arbitrary subsequence, the intervals that can be removed are determined as follows.

The heading interval Θ_i at the location p_i can be removed if a shorter connection goes through a different heading for each combination of the initial and final heading angles. So, Θ_i can be a part of the optimal solution only if the following equation holds

$$\exists\, \Theta_{i-w} \in H_{i-w}, \exists\, \Theta_{i+w} \in H_{i+w} : \\ \mathcal{L}_L(\Theta_{i-w}, \Theta_i) + \mathcal{L}_L(\Theta_{i-w}, \Theta_i) \leq \mathcal{L}_U(\Theta_{i-w}, \Theta_{i+w}). \tag{9}$$

With a slightly simplified notation, the heading interval Θ_i would not be part of the optimal solution if the lower bound value of the optimal solution using Θ_i is above the upper bound using another heading of the location being examined.

3.4 Computational Complexity

The computational complexity of the proposed BnB solution to the DTP is defined by the distance functions and determination of the upper and lower bounds. The time to compute all the distances can be bounded by $O(nk_{\max}^2)$, where $k_{\max}$ is the maximum number of heading intervals per a single location. All the upper bounds need to be computed for the specific window size w that can be bounded by $O(nwk_{\max}^2)$. Since w is fixed and we consider only three windows in this paper, we can bound the computation of all upper bounds by $O(nk_{\max}^2)$. The most demanding part is a determination of the lower bounds for the intervals bounding. We need to examine each combination of the initial, examined, and final heading angle as depicted in Fig. 3. Hence, the time complexity of `BoundIntervals(`$\mathcal{H}$`)` can be bounded by $O(nk_{\max}^3)$ which is also the complexity bound of Algorithm 1 for a fixed $\omega_{\max}$.

4 EMPIRICAL RESULTS

The feasibility of the proposed bounding has been empirically evaluated using randomly generated instances of the DTP using location density d similarly to [5]. In particular, 20 random instances with $n = 10$ locations are generated within the square box of the size $s = \sqrt{n}/d$ using uniform distribution and density $d = 0.3$. The evaluation is performed for turning radii $\rho \in \{0.2, 0.4, 0.6, 0.8, 1.0\}$. The sequence of locations is determined as a solution of the TSP using the Euclidean distance and the Lin-Kernighan-Helsgaun algorithm [7]. The proposed algorithm is implemented in the Julia programming language [2], version 1.6 with the available C++ implementation of the DIP [14]. All the presented results have been computed using the Intel processor i5-1035G1 running at up to 3.6 GHz. Each instance is solved once because the BnB algorithm is deterministic; however, the empirical results are reported as the median values of 20 random instances with the visualized 60 % nonparametric confidence interval. The maximal angular resolution $\omega_{\max} = 256$ is used for all the presented results.

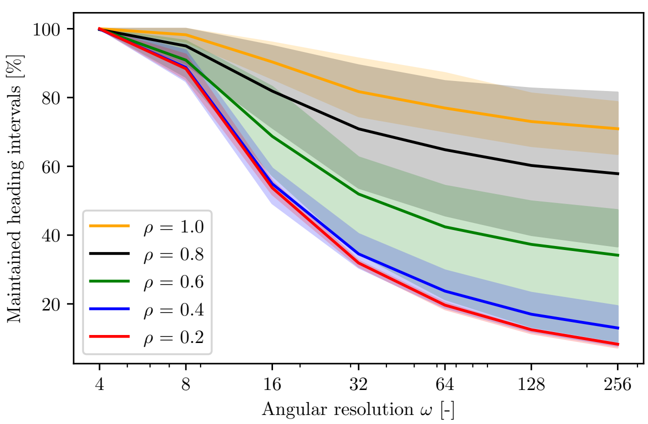


Figure 4: Percentage of the maintained intervals $\mathcal{H}$ with the increasing angular resolution ω.

The percentage of maintained intervals during the BnB-based solution of the DTP with respect to the actual angular resolution is depicted in Fig. 4 and corresponding real computational requirements are shown in Fig. 5. The results support the initial hypothesis on the significant reduction of the intervals that allows finding optimal solutions of the test instances of the DTP.

5 CONCLUSION

The proposed bounding unpromising heading intervals enables the BnB solution to the DTP by focusing the search towards the optimal solution. The bounding is based on the newly introduced maximization variant of the DIP (Max-DIP), for which its optimal solution is shown for instances satisfying the D4 assumption and with one side fixed heading angle. In the future, we aim to solve the introduced Max-DIP for the general case without the D4 assumption on the mutual distance of the goal locations that would allow solving any DTP instance optimality using the presented BnB.

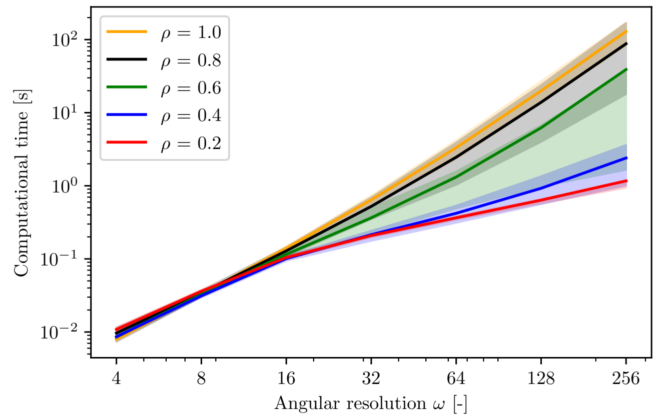


Figure 5: Real required computational time for solving the DTP instances with $n = 10$ locations depending on the angular resolutions.

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